I

Final Examination

Answer all questions. Each question carries ten points. You should justify your answer and show all details.

1. Let D be the region bounded by the curves $xy = 1, xy = 6, y = x$ and $y = 5x$ in the first quadrant. Evaluate the double integral
$$\iint_D e^{xy}\, dA(x,y) \ .$$

2. Consider the triple integral
$$\iiint_\Omega f(x,y,z)\, dV(x,y,z) \ ,$$
where Ω is the region bounded by $x^2+y^2+z^2 = 4,\ x+y+z = 2,\ x,y,z \geq 0$. Express it as (a) an integral in $dzdydx$ and (b) an integral in polar coordinates $d\rho d\varphi d\theta$.

3. Let Γ be the boundary of the triangle with vertices at $(0,0), (2,0)$ and $(0,3)$ in anticlockwise direction. Find the circulation of the flow $\mathbf{L} = (2x+y^2)\mathbf{i} + (2xy+x^2+\sin y)\mathbf{j}$ around Γ.

4. Let C be the intersection of the cylinder $x^2+y^2 = 4$ and the plane $x+2y+3z = 1$ oriented in anticlockwise direction. Find
$$\oint_C z\, dx + xy\, dy - 6\, dz \ .$$

5. Evaluate the surface integral
$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma \ ,$$
where S is the part of $z = x^2+y^2$ pinched between $z = 0, 4$ with normal pointing out and $\mathbf{F} = 3z\mathbf{i} + 5x\mathbf{j} - 2y\mathbf{k}$.

6. Find the work done by the force $\mathbf{E} = e^{yz}\mathbf{i} + (xze^{yz} + z\cos y)\mathbf{j} + (xye^{yz} + \sin y)\mathbf{k}$ on a person who walks from $A(1,0,0)$ to $B(0,1,5\pi/2)$ along the path $t \mapsto (\cos t, \sin t, t)$.

7. Let Ω be the set bounded by $z = 0, y = 0, y = 2$ and $z = 1 - x^2$ and S its boundary. Find the outward flux of the vector field
$$\mathbf{G}(x,y,z) = (x^2+\cos y)\mathbf{i} + (y + \sin xz)\mathbf{j} + (z + e^x)\mathbf{k}$$
across S.

8. Evaluate the improper integral
$$\int_0^\infty e^{-x^2} x^2\, dx \ .$$
You may use the formula
$$\int_{-\infty}^\infty e^{-x^2}\, dx = \sqrt{\pi}.$$

Exam I

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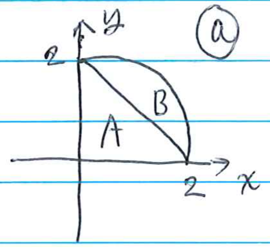
1. Let $u = xy,\ v = \frac{y}{x}$, $u \in [1,6],\ v \in [1,5]$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ -y/x^2 & 1/x \end{vmatrix} = \frac{2y}{x} = 2v$$

$$\therefore \iint_D e^{xy}\,dA = \int_1^5\int_1^6 e^u \frac{1}{2v}\,du\,dv = \frac{1}{2}\int_1^5\int_1^6 \frac{e^u}{v}\,du\,dv = \cdots.$$

2. 

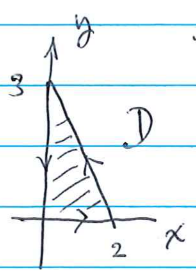
ⓐ over A, $2-x-y \le z \le \sqrt{4-x^2-y^2}$

over B, $0 \le z \le \sqrt{4-x^2-y^2}$

$$\therefore \iiint_\Omega f\,dV = \iint_A \int_{2-x-y}^{\sqrt{4-x^2-y^2}} f\,dz\,dA(x,y) + \iint_B \int_0^{\sqrt{4-x^2-y^2}} f\,dz\,dA(x,y)$$

$$= \int_0^2\int_0^{2-x}\int_{2-x-y}^{\sqrt{4-x^2-y^2}} f\,dz\,dy\,dx + \int_0^2\int_{2-x}^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}} f\,dz\,dy\,dx.$$

3. Use Green's thm, $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2y + 2x - 2y = 2x$



$$\therefore \oint_C (2x+y^2)\,dx + (2xy + x^2 + \sin y)\,dy = \iint_D 2x\,dA = \int_0^2\int_0^{\frac{3}{2}(2-x)} 2x\,dy\,dx = \cdots$$

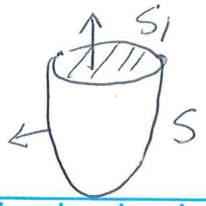
4. Use def. of line integral, $(x,y) \mapsto (x, y, \frac{1}{3}(1-x-2y))$

$\vec{r}: \theta \mapsto (2\cos\theta, 2\sin\theta, \frac{1}{3}(1-2\cos\theta-4\sin\theta))$

$\vec{r}\,' = (-2\sin\theta, 2\cos\theta, \frac{2}{3}\sin\theta - \frac{4}{3}\cos\theta)$

$$\oint_C z\,dx + xy\,dy - 6\,dz = \int_0^{2\pi} \Big[\tfrac{1}{3}(1-2\cos\theta-4\sin\theta)(-2\sin\theta) +$$
$$4\cos\theta\sin\theta\ 2\cos\theta$$
$$-6\left(\tfrac{2}{3}\sin\theta - \tfrac{4}{3}\cos\theta\right)\Big]\,d\theta$$

$$\doteq \frac{8\pi}{3}\ \#$$



5) Stokes' thm $\left(\iint\limits_S + \iint\limits_{S_1}\right) \nabla\times F\cdot\hat{n}\, d\sigma = 0$

On S_1, $\hat{n} = \hat{k}$.

$$\nabla\times\vec{F} = -2\hat{\imath}+3\hat{\jmath}+5\hat{k}$$

$$\nabla\times\vec{F}\cdot\hat{n} = (-2\hat{\imath}+3\hat{\jmath}+5\hat{k})\cdot\hat{k} = 5$$

$$\therefore \iint\limits_S \nabla\times F\cdot\hat{n} = -\iint\limits_{S_1} \nabla\times F\cdot\hat{n} = -5\iint\limits_{S_1} d\sigma = -5\times \text{area of } S_1 = -5\pi 4$$

$$= -20\pi \ \#$$

6)

$\vec{E}$ is conservative.

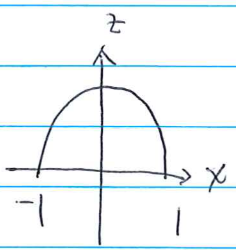
the potential:

$$\Phi = x e^{yz} + z\sin y$$

$$\therefore \int\limits_C \vec{E}\cdot d\vec{r} = \Phi(B)-\Phi(A)$$

$$= \frac{5\pi}{2}\sin 1 - 1.$$

7)



A cross section

$$\nabla\cdot\vec{G} = 2(x+1)$$

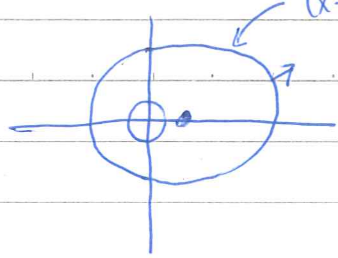
Divergence thm:

outward flux

$$= \iint\limits_S \vec{G}\cdot\hat{n}\, d\sigma$$

$$= \iiint\limits_\Omega \nabla\cdot\vec{G}\, dV = \iiint\limits_\Omega 2(x+1)\, dV$$

$$= \int_0^2\int_{-1}^{1}\int_0^{1-x^2} 2(x+1)\, dz\, dx\, dy = \cdots$$

9. 

Can't apply Green's thm to the region bdd by E. It is because $\vec{A}$ is not bdd at $(0,0)$.

We let C_r be a little circle around $(0,0)$.

By Green's thm

$(x,y)=(r\cos\theta, r\sin\theta)$

$(x',y')=(-r\sin\theta, r\cos\theta)$

$|(x',y')|=r$

$\hat{n}=(\cos\theta, \sin\theta)$

$$\int_E \vec{A}\cdot\hat{n}\,ds = \int_{C_r} \vec{A}\cdot\hat{n}\,ds$$

$$= \int_0^{2\pi}\left(\frac{r\cos\theta}{r^2}\hat{i}+\frac{r\sin\theta}{r^2}\hat{j}\right)\cdot\hat{n}\; r\,d\theta$$

$$= 2\pi.$$

10. (a) If $F_j = \dfrac{\partial\Phi}{\partial x_j}$, then

$$\frac{\partial F_j}{\partial x_i} = \frac{\partial^2\Phi}{\partial x_i\partial x_j} = \frac{\partial^2\Phi}{\partial x_j\partial x_i} = \frac{\partial F_i}{\partial x_j} \quad \#$$

(b)

$$\frac{\partial}{\partial x_j}\Phi(x) = \int_0^1 \frac{\partial}{\partial x_j}\left[F_1(t\vec{x})x_1 + F_2(t\vec{x})x_2 + \cdots + F_n(t\vec{x})x_n\right]dt$$

$$= \int_0^1\Big[\frac{\partial F_1}{\partial x_j}(t\vec{x})\,t x_1 + \frac{\partial F_2}{\partial x_j}(t\vec{x})\,t x_2 + \cdots + \frac{\partial F_n}{\partial x_j}(t\vec{x})\,x_n$$

$$+ F_j(t\vec{x})\Big]dt$$

$$= \int_0^1 \sum_i\left(\frac{\partial F_i}{\partial x_j}(t\vec{x})\,x_i\right)t + F_j(t\vec{x})\,dt$$

$$= \int_0^1\left(\sum_i \frac{\partial F_j}{\partial x_i}(t\vec{x})\,x_i\right)t + F_j(t\vec{x})\,dt$$

$$= \int_0^1 \frac{d}{dt}\left(F_j(t\vec{x})\,t\right)dt$$

$$= F_j(t\vec{x})\,t\Big|_{t=0}^{t=1} = F_j(\vec{x}).$$